Quiz 8.4/Practice Problems 8.7

# 1 Quiz Problem 1

1. Calculate $\int \frac{-x}{x^3-3x^2+2x}\,dx$.

We'll decompose $\frac{-x}{x^3-3x^2+2x}$ to make it easier to integrate. First we factor the top and bottom and cancel common factors:

$$\frac{-x}{x(x^2-3x+2)} = \frac{-x}{x(x-2)(x-1)} = \frac{-1}{(x-2)(x-1)}$$

Now we set up our decomposition:

$$\frac{-1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

Now multiply by the denominator $(x-2)(x-1)$ on both sides:

$$-1 = A(x-1) + B(x-2)$$

Subbing in $x = 1$ yields the equation $-1 = B(-1)$ so $B = 1$, and subbing in $x = 2$ yields $A = -1$. Thus

$$\int \frac{-x}{x^3-3x^2+2x}\,dx = \int \frac{-1}{x-2} + \frac{1}{x-1}\,dx = -\ln|x-2| + \ln|x-1| + C = \ln|\tfrac{x-1}{x-2}| + C$$

# 2 OR Quiz Problem 2

2. Calculate $\int \frac{1}{(x^2+1)(x-2)}\,dx$

We'll decompose $\frac{1}{(x^2+1)(x-2)}$. Note the top and bottom are already factored, so we'll set up our decomposition:

$$\frac{1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

Now multiply by the denominator $(x^2+1)(x-2)$ on both sides:

$$1 = (Ax+B)(x-2) + C(x^2+1)$$

Setting $x = 2$ yields $1 = 5B$ so $C = 1/5$. To find $A$, $B$, look at the highest and lowest degree terms on the right.

The coefficient of $x^2$ on the right is $A + B$, so equating this with the coefficient of $x^2$ on the left (there is none), we have $A + C = 0$, so $A = -C = -1/5$.

The constant term on the right is $-2B + C$, so equating this with the constant term on the left, we have $-2B + C = 1$. Thus $-2B + 1/5 = 1$, so $-2B = 4/5$, so $B = -2/5$.

Then,

$$\int \frac{1}{(x^2+1)(x-2)}\,dx = \int \frac{-x/5 - 2/5}{x^2+1} + \frac{1/5}{x-2}\,dx = -\frac{1}{5}\int \frac{x+2}{x^2+1}\,dx + \frac{1}{5}\int \frac{1}{x-2}\,dx \quad (1)$$

$$= -\frac{1}{5}\int \frac{x}{x^2+1}\,dx - \frac{1}{5}\int \frac{2}{x^2+1}\,dx + \frac{1}{5}\int \frac{1}{x-2}\,dx \quad (2)$$

$$= -\frac{1}{5}\int \frac{x}{x^2+1}\,dx - \frac{2}{5}\arctan(x) + \frac{1}{5}\ln|x-2| + C \quad (3)$$

To do the integral $\int -\frac{1}{5}\int \frac{x}{x^2+1}\,dx$ we'll do a u-sub $u = x^2+1$, so $du = 2x\,dx$, or $du/2 = x\,dx$. Then $-\frac{1}{5}\int \frac{x}{x^2+1} = -\frac{1}{10}\int \frac{1}{u}\,du = (-1/10)\ln|u| + C = (-1/10)\ln|x^2+1| + C$, so:

$\int \frac{1}{(x^2+1)(x-2)}\,dx = -\frac{1}{5}\int \frac{x}{x^2+1}\,dx - \frac{1}{5}\int \frac{2}{x^2+1}\,dx + \frac{1}{5}\int \frac{1}{x-2}\,dx = (-1/10)\ln(x^2+1) - \frac{2}{5}\arctan(x) + \frac{1}{5}\ln|x-2| + C$ and we're done.

## 3 Practice Problems:

## 4 Problem 1

Determine whether $\int_0^\pi \csc^2 x\,dx$ converges. If it does, determine the value of the integral.

Note that $\csc^2 x = 1/\sin^2 x$ and $\sin x = 0$ at 0 and $\pi$. Thus, we are dealing with an indefinite integral. Since there's a problem at both ends, fix $0 < D < \pi$. We'll show that neither $\int_0^D \csc^2 x\,dx$ nor $\int_D^\pi \csc^2 x\,dx$ converge, so that $\int_0^\pi \csc^2 x\,dx$ cannot either:

$\int_a^b f\,dx$ converges $\iff$ there's a number $D$ $(a < D < b)$ so $\int_a^D f\,dx$ and $\int_D^b f\,dx$ both converge.

First we show $\int_0^D \csc^2 x\,dx$ doesn't converge:

$$\begin{aligned}\int_0^D \csc^2 x\,dx = \lim_{a\to 0^+}\int_a^D \csc^2 x\,dx &= \lim_{a\to 0^+}[-\ln|\csc x + \cot x|]_a^D &(4)\\ &= \lim_{a\to 0^+} -\ln|\csc D + \cot D| - [-\ln|\csc a + \cot a|] &(5)\\ &= \lim_{a\to 0^+} \ln|\frac{1+\cos a}{\sin a}| - \ln|\csc D + \cot D| &(6)\\ &= \ln(\lim_{a\to 0^+}|\frac{1+\cos a}{\sin a}|) - \ln|\csc D + \cot D| = \infty &(7)\\ & &(8)\end{aligned}$$

Since $\lim_{a\to 0^+}|\frac{1+\cos a}{\sin a}| = \infty$ and $\lim_{x\to\infty}\ln x = \infty$.

Now we show $\int_D^\pi \csc^2 x\,dx$ doesn't converge. Well,

$$\begin{aligned}\int_D^\pi \csc^2 x\,dx = \lim_{b\to\pi^-}\int_D^b \csc^2 x\,dx &= \lim_{b\to\pi^-}[-\ln|\csc x + \cot x|]_D^b &(9)\\ &= \lim_{b\to\pi^-} -\ln|\csc b + \cot b| - [-\ln|\csc D + \cot D|] &(10)\\ &= \lim_{b\to\pi^-} \ln|\csc D + \cot D| - \ln|\frac{1+\cos b}{\sin b}| &(11)\\ &= \ln|\csc D + \cot D| - \ln(\lim_{b\to\pi^-}|\frac{1+\cos b}{\sin b}|) &(12)\end{aligned}$$

Note $\lim_{b\to\pi^-}\frac{1+\cos b}{\sin b} = \frac{0}{0}$, so we apply L'Hopital's rule: $\lim_{b\to\pi^-}\frac{1+\cos b}{\sin b} = \lim_{b\to\pi^-}\frac{-\sin b}{\cos b} = 0$.

Thus $\lim_{b\to\pi^-}\ln|\frac{1+\cos b}{\sin b}| = \lim_{c\to 0^+}\ln c = -\infty$.

Therefore,

$$\ln|\csc D + \cot D| - \ln(\lim_{b\to\pi^-}|\frac{1+\cos b}{\sin b}|) = \infty.$$

Thus $\int_0^\pi \csc^2 x\,dx$ diverges, since $\int_0^D \csc^2 x\,dx = \infty = \int_D^\pi \csc^2 x\,dx$ for any $a < D < b$.

## 5 Problem 2

Determine whether $\int_0^1 \frac{3x^2-1}{x^3-x}\,dx$ converges. If it does, determine the value of the integral.

Note that $\frac{3x^2-1}{x^3-x}$ has asymptotes at $x = \pm 1$, and $x = 0$. Since $x = 0$ and $x = 1$ are in our bounds, we are dealing with an improper integral. Let's turn it into limits:

$$\int_0^1 \frac{3x^2-1}{x^3-x}\,dx = \lim_{a\to 0^+}\lim_{b\to 1^-}\int_a^b \frac{3x^2-1}{x^3-x}\,dx \tag{13}$$

Note also that $\int \frac{3x^2-1}{x^3-x}\,dx$ is solved by a u-sub if $u = x^3 - x$, $du = 3x^2 - 1\,dx$, so $\int \frac{3x^2-1}{x^3-x}\,dx = \int \frac{1}{u}\,du = \ln(x^3 - x) + C$.

Fix $0 < D < 1$. We'll show that neither $\int_0^D \frac{3x^2-1}{x^3-x}\,dx$ nor $\int_D^1 \frac{3x^2-1}{x^3-x}\,dx$ converge, so that $\int_0^1 \frac{3x^2-1}{x^3-x}\,dx$ cannot either ($\int_a^b f\,dx$ converges $\iff$ there's a number $d$ ($a < d < b$) so $\int_a^d f\,dx$ and $\int_d^b f\,dx$ both converge).

$$\begin{aligned}
\int_0^D \frac{3x^2-1}{x^3-x}\,dx &= \lim_{a\to 0^+}\int_a^D \frac{3x^2-1}{x^3-x}\,dx && (14)\\
&= \lim_{a\to 0^+}\left[\ln|x^3-x|\right]_a^D && (15)\\
&= \lim_{a\to 0^+} \ln|D^3-D| - \ln|a^3-a| = \infty && (16)
\end{aligned}$$

Since $\lim_{a\to 0^+} -\ln|a^3-a| = \lim_{c\to 0^+} -\ln c = \infty$. Similarly one shows that $\lim_{b\to 1^-}\int_D^b \frac{3x^2-1}{x^3-x}\,dx$ does not converge:

$$\begin{aligned}
\int_D^1 \frac{3x^2-1}{x^3-x}\,dx &= \lim_{b\to 1^-}\int_D^b \frac{3x^2-1}{x^3-x}\,dx && (17)\\
&= \lim_{b\to 1^-}\left[\ln|x^3-x|\right]_D^b && (18)\\
&= \lim_{b\to 1^-} \ln|b^3-b| - \ln|D^3-D| = -\infty && (19)
\end{aligned}$$

## 6 Problem 3

Determine whether $\int_0^1 \frac{e^t}{\sqrt{e^t-1}}\,dt$ converges. If it does, determine the value of the integral.

Note the integrand only has a singularity at the endpoint $t = 0$. since $\sqrt{e^0 - 1} = \sqrt{0} = 0$. Then we're dealing with an improper integral:

$$\int_0^1 \frac{e^t}{\sqrt{e^t-1}}\,dt = \lim_{a\to 0^+}\int_a^1 \frac{e^t}{\sqrt{e^t-1}}\,dt \tag{20}$$

Note also that the integral $\int \frac{e^t}{\sqrt{e^t-1}}\,dt$ is solved by a u-sub if $u = e^t - 1$, so $du = e^t\,dt$:

$\int \frac{e^t}{\sqrt{e^t-1}}\,dt = \int \frac{1}{\sqrt{u}}\,dt = 2\sqrt{u} + C = 2\sqrt{e^t-1} + C$

Then, we have

$$\begin{aligned}\int_0^1 \frac{e^t}{\sqrt{e^t - 1}}\, dt &= \lim_{a \to 0^+} \int_a^1 \frac{e^t}{\sqrt{e^t - 1}}\, dt && (21)\\ &= \lim_{a \to 0^+} [2\sqrt{e^t - 1}]_a^1 && (22)\\ &= \lim_{a \to 0^+} 2\sqrt{e^1 - 1} - 2\sqrt{e^a - 1} = 2\sqrt{e - 1} && (23)\end{aligned}$$

and so $\int_0^1 \frac{e^t}{\sqrt{e^t-1}}\, dt = 2\sqrt{e - 1}$